NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

N65-89048

~~X64 11805~~ *

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Code 2A

PROPOSED JOURNAL ARTICLE

(NASA TMX 51461)

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31 Oct. 1963 refs

Submitted for Publication

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Prepared for

The Journal of American Institute of Aeronautics and Astronautics

October 31, 1963

E-2345

# APPROXIMATION OF THE EIGENVALUES FOR HEAT TRANSFER IN LAMINAR TUBE SLIP FLOW

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For convective heat transfer in laminar continuum tube flow with uniform wall heat flux, Sellars et al[1] have obtained asymptotic formulas for the eigenvalues and coefficients through a generalization of constant wall temperature results. An improved and more direct treatment has been presented by Dzung[2].

The advent of space flight has brought about increased interest in the heat transfer to low-density gas flow in tubes. Sparrow and Lin[3] have considered the fully developed heat transfer in circular tubes under slip-flow conditions. It is of interest to determine the possible application of the method of Sellars et al to laminar tube slip flow.

We wish to find solutions of

$$(d/d\eta)[\eta(dR_n/d\eta)] + \lambda_n(2f)\eta R_n = 0 \tag{1}$$

subject to

$$R_n(0) = 1 \tag{2}$$

and

$$[(dR_n/d\eta)] = 0 \quad \text{at} \quad \eta = 0 \quad \text{and} \quad \eta = 1 \tag{3}$$

where $\eta$ is the dimensionless radial distance $(r/r_0)$, $f(\eta)$ is the dimensionless velocity distribution,

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$$f(\eta) \equiv u(\eta)/\bar{u} = 2[1 - \eta^2 + 4\alpha]/[1 + 8\alpha] \tag{4}$$

and $\alpha \equiv (\xi_u/d)$. The function $R_n$ represents the radial temperature distribution in the thermal entrance region, and $\lambda_n$ is the eigenvalue. $\xi_u$ is the velocity slip coefficient[3]. The velocity distribution as given in Eq. (4) assumes that thermal creep is negligible.

In accordance with the method of Sellars et al., a solution of the form

$$R_n(\eta) = \exp[g(\eta)] \tag{5}$$

is considered, where

$$g = \sqrt{\lambda}\, g_0 + g_1 + (g_2/\sqrt{\lambda}) + \ldots \tag{6}$$

and, since $\lambda$ is assumed to be large, only the first two terms of the above series are retained. It can be shown that $R_n$ is given from Eqs. (5) and (6) as

$$R_n = \left\{A \exp\left[i\sqrt{\lambda_n}\int_0^\eta (2f)^{1/2}\, d\eta\right] + B \exp\left[-i\sqrt{\lambda_n}\int_0^\eta (2f)^{1/2}\, d\eta\right]\right\} \bigg/ \eta^{1/2}(2f)^{1/4} \tag{7}$$

excluding the singular point $\eta = 0$. It should be noted that, for continuum flow ($\alpha = 0$), a singularity also exists at $\eta = 1$, since $[f(1)]_{\alpha=0} = 0$. This has required the developement of an alternate solution valid near $\eta = 1$[1,2]. For slip flow, no singularity exists at $\eta = 1$, since $[f(1)]_{\alpha\neq 0} = 8\alpha/(1 + 8\alpha) = u_s/\bar{u}$, where $\bar{u}_s$ is the slip velocity.

The coefficients A and B are determined from continuation of Eq. (7) to the central zone $\eta \approx 0$, where $R_n(\eta)$ can be approximated by a Bessel function

$$R_n(\eta) \approx J_0(\sqrt{2\lambda_n f(0)}\ \eta) \qquad \eta^2 \ll 1 \tag{8}$$

where $f(0) = 2(1 + 4\alpha)/(1 + 8\alpha) = u_c/\bar{u}$. From the asymptotic expression of Bessel functions, the coefficients are determined, so that

$$R_n(\eta) = (\pi\eta)^{-1/2}(\overset{2/\lambda_n f}{\cancel{2\lambda_n/f}})^{1/4} \cos\left[\sqrt{\lambda_n}\ I - \pi/4\right] \tag{9}$$

where

$$I \equiv \int_0^\eta (2f)^{1/2}\, d\eta = \left[\eta\sqrt{1 + 4\alpha - \eta^2} + (1 + 4\alpha)\,\text{arc sin}(\eta/\sqrt{1 + 4\alpha})\right]\Big/\sqrt{1 + 8\alpha} \tag{10}$$

The slope of $R_n(\eta)$ at the wall is found by differentiating Eq. (9) and setting $\eta = 1$; the result is

$$R_n'(1) = (8\pi)^{-1/2}(1 + 8\alpha)^{1/4}(4\alpha)^{-5/4}\lambda_n^{-1/4}[(E + F\gamma_n)\cos\gamma_n + (E - F\gamma_n)\sin\gamma_n] \tag{11}$$

where $\gamma_n \equiv \sqrt{\lambda_n}\ I_1$, $I_1 \equiv \int_0^1 (2f)^{1/2}\, d\eta$, $E \equiv 1 - 4\alpha$, and

$F \equiv 4(4\alpha)^{3/2}\Big/\left[\sqrt{4\alpha} + (1 + 4\alpha)\,\text{arc sin}(1/\sqrt{1 + 4\alpha})\right]$.

Setting $R_n'(1) = 0$ yields a series of eigenvalues $\lambda_n$ with the corresponding eigenfunction $R_n$ as roots of the equation

$$\tan\gamma_n = (F\gamma_n + E)/(F\gamma_n - E) \tag{12}$$

The coefficients $c_n$ of the series expansion for uniform wall heat flux are determined by the requirement that[3,4]

$$\sum_{n=1}^{\infty} c_n R_n(\eta) = -\left[\left(\eta^2 - \frac{1}{4}\eta^4 - \frac{7}{24}\right) - \left(\frac{1}{2}\eta^2 - \frac{1}{4}\eta^4\right)\left(\frac{u_s}{\bar{u}}\right) + \left(\frac{1}{24}\right)\left(\frac{u_s}{\bar{u}}\right)^2\right] \tag{13}$$

which, with the orthogonality property of the eigenfunctions, leads to

$$c_n = 1\Big/[\lambda(\partial^2 R/\partial\eta\,\partial\lambda)]_{\eta=1,\lambda=\lambda_n} \tag{14}$$

Hence,

$$D_n \equiv c_n R_n(1) = -(16\alpha)\Big/\left[E + (E^2/F) + F\gamma_n^2\right] \tag{15}$$

These expressions were derived on the assumption that $\lambda_n$ is large and consequently are supposedly valid only in that limit. However, the use of Eqs. (12) and (15) gives values that appear to "fit" between results for continuum flow $(u_s = 0)$ and for slug flow $(u_s/\bar{u} \to 1)$ even for the values of $n$ as small as 2, as can be seen from the comparison shown in Table I. The results for continuum flow were obtained from expressions presented by Dzung[2], while for slug flow, the eigenvalues are obtained as the roots of $J_1(\sqrt{2\lambda_n}) = 0$, where $J_1$ is a Bessel function of the first kind and of first order; the coefficients $D_n$ are then obtained from the simple result $D_n = -1/\lambda_n$.

It should be mentioned that the first few eigenvalues apparently become slightly inaccurate (i.e., do not fit) as $(u_s/\bar{u} \to 0)$. Also, interactions between the velocity and temperature fields, such as thermal creep, have been neglected, as noted earlier.

From this comparison it appears that the method of Sellars, et al. has definite application to determining the eigenvalues and coefficients for heat transfer in laminar tube slip flow. The author is currently extending the treatment outlined here to the problem of heat transfer to laminar slip flow in a parallel plate channel.

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TABLE I

| | $u_s/\bar{u} = 0$ | 2/5 | 2/3 | 1 |
|---|---|---|---|---|
| $\sqrt{\lambda_1}$ | 2.531 | ------- | 2.65 | 2.710 |
| $\sqrt{\lambda_2}$ | 4.578 | 4.71 | 4.75 | 4.955 |
| $\sqrt{\lambda_3}$ | 6.599 | 6.78 | 6.88 | 7.195 |
| $\sqrt{\lambda_4}$ | 8.610 | 8.81 | 9.00 | 9.425 |
| $D_1$ | -0.1985 | ------- | -0.1670 | -0.1360 |
| $D_2$ | -0.0693 | -0.0594 | -0.0515 | -0.0406 |
| $D_3$ | -0.0365 | -0.0306 | -0.0247 | -0.0194 |
| $D_4$ | -0.0230 | -0.0217 | -0.0145 | -0.0113 |